**Scholarly Research Journal for Interdisciplinary Studies,**
*Online ISSN 2278-8808, SJIF 2018 = 6.371, www.srjis.com*
*PEER REVIEWED JOURNAL, SEPT-OCT, 2018, VOL- 6/47*

# HARMONIC MAPPING; CONCEPT AND APPLICATIONS

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***Abstract***

*In the mathematical field of differential geometry, a smooth map from one Riemannian manifold to another Riemannian manifold is called harmonic if its coordinate representatives satisfy a certain nonlinear partial differential equation. This partial differential equation for a mapping also arises as the Euler-Lagrange equation of a functional generalizing the Dirichlet energy (which is often itself called "Dirichlet energy"). As such, the theory of harmonic maps encompasses both the theory of unit-speed geodesics in Riemannian geometry, and the theory of harmonic functions on open subsets of Euclidean space and on Riemannian manifolds. Informally, the Dirichlet energy of a mapping f from a Riemannian manifold M to a Riemannian manifold N can be thought of as the total amount that f "stretches" M in allocating each of its elements to a point of N. For instance, a rubber band which is stretched around a (smooth) stone can be mathematically formalized as a mapping from the points on the unstretched band to the surface of the stone. The unstretched band and stone are given Riemannian metrics as embedded submanifolds of three-dimensional Euclidean space; the Dirichlet energy of such a mapping is then a formalization of the notion of the total tension involved. Harmonicity of such a mapping means that, given any hypothetical way of physically deforming the given stretch, the tension (when considered as a function of time) has first derivative zero when the deformation begins. The techniques used by Richard Schoen and Uhlenbeck to study the regularity theory of harmonic maps have likewise been the inspiration for the development of many analytic methods in geometric theory.*



## Introduction

The theory of harmonic maps was initiated in 1964 by James Eells and Joseph Sampson, who showed that in certain geometric contexts, arbitrary smooth maps could be deformed into harmonic maps. Their work was the inspiration for Richard Hamilton's first work on the Ricci flow. Harmonic maps and the associated harmonic map heat flow, in and of themselves, are among the most widely studied topics in the field of geometric analysis.

The discovery of the "bubbling" of sequences of harmonic maps, due to Jonathan Sacks and Karen Uhlenbeck, has been particularly influential, as the same phenomena has been found in many other geometric contexts. Notably, Uhlenbeck's parallel discovery of bubbling of Yang–Mills fields is important in Simon Donaldson's work on four-dimensional

manifolds, and Mikhael Gromov's later discovery of bubbling of pseudoholomorphic curves is significant in applications to symplectic geometry and quantum cohomology.

**Mathematical Meaning**

Here the notion of the laplacian of a map is considered from three different perspectives. A map is called harmonic if its laplacian vanishes; it is called totally geodesic if its hessian vanishes.

***Integral formulation***

Let $(M, g)$ and $(N, h)$ be Riemannian manifolds. Given a smooth map $f$ from $M$ to $N$, the pullback $f^*h$ is a symmetric 2-tensor on $M$; the **energy density** $e(f)$ of $f$ is one-half of its $g$-trace. If $M$ is oriented and $M$ is compact, the **Dirichlet energy** of $f$ is defined as, where $d\mu_g$ is the volume form on $M$ induced by $g$. Even if $M$ is noncompact, the following definition is meaningful: the **Laplacian** or **tension field** $\Delta f$ of $f$ is the vector field in $N$ along $f$ such that for any one-parameter family of maps $f_s : M \to N$ with $f_0 = f$ and for which there exists a precompact open set $K$ of $M$ such that $f_s|_{M-K} = f|_{M-K}$ for all $s$; one supposes that the parametrized family is smooth in the sense that the associated map $(-\varepsilon, \varepsilon) \times M \to N$ given by $(s, p) \mapsto f_s(p)$ is smooth.In case $M$ is compact, the Laplacian of $f$ can then be thought of as the gradient of the Dirichlet energy functional.

***Local coordinates***

Let $U$ be an open subset of $\mathbb{R}^m$ and let $V$ be an open subset of $\mathbb{R}^n$. For each $i$ and $j$ between 1 and $n$, let $g_{ij}$ be a smooth real-valued function on $U$, such that for each $p$ in $U$, one has that the $m \times m$ matrix $[g_{ij}(p)]$ is symmetric and positive-definite. For each $\alpha$ and $\beta$ between 1 and $m$, let $h_{\alpha\beta}$ be a smooth real-valued function on $V$, such that for each $q$ in $V$, one has that the $n \times n$ matrix $[h_{\alpha\beta}(q)]$ is symmetric and positive-definite. Denote the inverse matrices by $[g^{ij}(p)]$ and $[h^{\alpha\beta}(q)]$.

For each $i, j, k$ between 1 and $n$ and each $\alpha, \beta, \gamma$ between 1 and $m$ define the Christoffel symbols $\Gamma(g)^k_{ij} : U \to \mathbb{R}$ and $\Gamma(h)^\gamma_{\alpha\beta} : V \to \mathbb{R}$

Given a smooth map $f$ from $U$ to $V$, its **hessian** defines for each $i$ and $j$ between 1 and $m$ and for each $\alpha$ between 1 and $n$ the real-valued function $\nabla(df)^\alpha_{ij}$ on $U$ by

Its **laplacian** or **tension field** defines for each $\alpha$ between 1 and $n$ the real-valued function $(\Delta f)^\alpha$ on $U$.

***Bundle formalism***

Let $(M, g)$ and $(N, h)$ be Riemannian manifolds. Given a smooth map $f$ from $M$ to $N$, one can consider its differential $df$ as a section of the vector bundle $T^*M \otimes f^*TN$ over $M$; all this says is that for each $p$ in $M$, one has a linear map $df_p$ as $T_pM \to T_{f(p)}N$. The Riemannian metrics on $M$ and $N$ induce a bundle metric on $T^*M \otimes f^*TN$, and so one may define $1/2 \,|\, df \,|^2$ as a smooth function on $M$, known as the energy density.

The bundle $T^*M \otimes f^*TN$ also has a metric-compatible connection induced from the Levi-Civita connections on $M$ and $N$. So one may take the covariant derivative $\nabla(df)$, which is a section of the vector bundle $T^*M \otimes T^*M \otimes f^*TN$ over $M$; this says that for each $p$ in $M$, one has a bilinear map $(\nabla(df))_p$ as $T_pM \times T_pM \to T_{f(p)}N$. This section is known as the hessian of $f$.

Using $g$, one may trace the hessian of $f$ to arrive at the laplacian or tension field of $f$, which is a section of the bundle $f^*TN$ over $M$; this says that the laplacian of $f$ assigns to each $p$ in $M$ an element of $T_{f(p)}N$. It is defined by

where $e_1, ..., e_m$ is a $g_p$-orthonormal basis of $T_pM$.

**Examples of harmonic maps**

Let $(M, g)$ and $(N, h)$ be smooth Riemannian manifolds. The notation $g_{\text{stan}}$ is used to refer to the standard Riemannian metric on Euclidean space.

- Every totally geodesic map $(M, g) \to (N, h)$ is harmonic; this follows directly from the above definitions. As special cases:
  - For any $q$ in $N$, the constant map $(M, g) \to (N, h)$ valued at $q$ is harmonic.
  - The identity map $(M, g) \to (M, g)$ is harmonic.

- If $f : M \to N$ is an immersion, then $f : (M, f^*h) \to (N, h)$ is harmonic if and only if $f$ is minimal relative to $h$. As a special case:
  - If $f : \mathbb{R} \to (N, h)$ is a constant-speed immersion, then $f : (\mathbb{R}, g_{\text{stan}}) \to (N, h)$ is harmonic if and only if $f$ solves the geodesic differential equation.

  Recall that if $M$ is one-dimensional, then minimality of $f$ is equivalent to $f$ being geodesic, although this does not imply that it is a constant-speed parametrization, and hence does not imply that $f$ solves the geodesic differential equation.
- A smooth map $f : (M, g) \to (\mathbb{R}^n, g_{\text{stan}})$ is harmonic if and only if each of its $n$ component functions are harmonic as maps $(M, g) \to (\mathbb{R}, g_{\text{stan}})$. This coincides with the notion of harmonicity provided by the Laplace-Beltrami operator.

- Every holomorphic map between Kahler manifolds is harmonic.
- Every harmonic morphism between Riemannian manifolds is harmonic.

**Harmonic map heat flow**

Let $(M, g)$ and $(N, h)$ be smooth Riemannian manifolds. A harmonic map heat flow on an interval $(a, b)$ assigns to each $t$ in $(a, b)$ a twice-differentiable map $f_t : M \to N$ in such a way that, for each $p$ in $M$, the map $(a, b) \to N$ given by $t \mapsto f_t(p)$ is differentiable, and its derivative at a given value of $t$ is, as a vector in $T_{f_t(p)}N$, equal to $(\Delta f_t)_p$. This is usually abbreviated as:

Eells and Sampson introduced the harmonic map heat flow and proved the following fundamental properties:

- Regularity. Any harmonic map heat flow is smooth as a map $(a, b) \times M \to N$ given by $(t, p) \mapsto f_t(p)$.

  Now suppose that $M$ is a closed manifold and $(N, h)$ is geodesically complete.
- Existence. Given a continuously differentiable map $f$ from $M$ to $N$, there exists a positive number $T$ and a harmonic map heat flow $f_t$ on the interval $(0, T)$ such that $f_t$ converges to $f$ in the $C^1$ topology as $t$ decreases to 0.
- Uniqueness. If $\{ f_t : 0 < t < T \}$ and $\{ f_t : 0 < t < T \}$ are two harmonic map heat flows as in the existence theorem, then $f_t = f_t$ whenever $0 < t < \min(T, T)$.

As a consequence of the uniqueness theorem, there exists a maximal harmonic map heat flow with initial data $f$, meaning that one has a harmonic map heat flow $\{ f_t : 0 < t < T \}$ as in the statement of the existence theorem, and it is uniquely defined under the extra criterion that $T$ takes on its maximal possible value, which could be infinite.

**Eells and Sampson's theorem**

The primary result of Eells and Sampson's theorem is the following:

Let $(M, g)$ and $(N, h)$ be smooth and closed Riemannian manifolds, and suppose that the sectional curvature of $(N, h)$ is nonpositive. Then for any continuously differentiable map $f$ from $M$ to $N$, the maximal harmonic map heat flow $\{ f_t : 0 < t < T \}$ with initial data $f$ has $T = \infty$, and as $t$ increases to $\infty$, the maps $f_t$ subsequentially converge in the $C^\infty$ topology to a harmonic map.

In particular, this shows that, under the assumptions on $(M, g)$ and $(N, h)$, every continuous map is homotopic to a harmonic map. The very existence of a harmonic map in each homotopy class, which is implicitly being asserted, is part of the result. In 1967, Philip

Hartman extended their methods to study uniqueness of harmonic maps within homotopy classes, additionally showing that the convergence in the Eells-Sampson theorem is strong, without the need to select a subsequence. Eells and Sampson's result was adapted to the setting of the Dirichlet boundary value problem, when $M$ is instead compact with nonempty boundary, by Richard Hamilton in 1975.

For many years after Eells and Sampson's work, it was unclear to what extent the sectional curvature assumption on $(N, h)$ was necessary. Following the work of Kung-Ching Chang, Wei-Yue Ding, and Rugang Ye in 1992, it is widely accepted that the maximal time of existence of a harmonic map heat flow cannot "usually" be expected to be infinite. Their results strongly suggest that there are harmonic map heat flows with "finite-time blowup" even when both $(M, g)$ and $(N, h)$ are taken to be the two-dimensional sphere with its standard metric. Since elliptic and parabolic partial differential equations are particularly smooth when the domain is two dimensions, the Chang-Ding-Ye result is considered to be indicative of the general character of the flow.

**Bochner formula and rigidity**

The main computational point in the proof of Eells and Sampson's theorem is an adaptation of the Bochner formula to the setting of a harmonic map heat flow $\{ f_t : 0 < t < T \}$. This formula says

This is also of interest in analyzing harmonic maps themselves; suppose $f : M \to N$ is harmonic. Any harmonic map can be viewed as a constant-in-$t$ solution of the harmonic map heat flow, and so one gets from the above formula that

If the Ricci curvature of $g$ is positive and the sectional curvature of $h$ is nonpositive, then this implies that $\Delta e(f)$ is nonnegative. If $M$ is closed, then multiplication by $e(f)$ and a single integration by parts shows that $e(f)$ must be constant, and hence zero; hence $f$ must itself be constant. Richard Schoen & Shing-Tung Yau (1976) note that this can be extended to noncompact $M$ by making use of Yau's theorem asserting that nonnegative subharmonic functions which are $L^2$-bounded must be constant. In summary, according to Eells & Sampson (1964) and Schoen & Yau (1976), one has:

Let $(M, g)$ and $(N, h)$ be smooth and complete Riemannian manifolds, and let $f$ be a harmonic map from $M$ to $N$. Suppose that the Ricci curvature of $g$ is positive and the sectional curvature of $h$ is nonpositive.

- If *M* and *N* are both closed then *f* must be constant.
- If *N* is closed and *f* has finite Dirichlet energy, then it must be constant.

In combination with the Eells-Sampson theorem, this shows (for instance) that if (*M*, *g*) is a closed Riemannian manifold with positive Ricci curvature and (*N*, *h*) is a closed Riemannian manifold with nonpositive sectional curvature, then every continuous map from *M* to *N* is homotopic to a constant.

The general idea of deforming a general map to a harmonic map, and then showing that any such harmonic map must automatically be of a highly restricted class, has found many applications. For instance, Yum-Tong Siu (1980) found an important complex-analytic version of the Bochner formula, asserting that a harmonic map between Kahler manifolds must be holomorphic, provided that the target manifold has appropriately negative curvature. As an application, by making use of the Eells-Sampson existence theorem for harmonic maps, he was able to show that if (*M*, *g*) and (*N*, *h*) are smooth and closed Kähler manifolds, and if the curvature of (*N*, *h*) is appropriately negative, then *M* and *N* must be biholomorphic or anti-biholomorphic if they are homotopic to each other; the biholomorphism (or anti-biholomorphism) is precisely the harmonic map produced as the limit of the harmonic map heat flow with initial data given by the homotopy. By an alternative formulation of the same approach, Siu was able to prove a variant of the still-unsolved Hodge conjecture, albeit in the restricted context of negative curvature.

**Applications**

- If, after applying the rubber *M* onto the marble *N* via some map, one "releases" it, it will try to "snap" into a position of least tension. This "physical" observation leads to the following mathematical problem: given a homotopy class of maps from *M* to *N*, does it contain a representative that is a harmonic map?
- Existence results on harmonic maps between manifolds has consequences for their curvature.
- Once existence is known, how can a harmonic map be constructed explicitly? (One fruitful method uses twistor theory)
- In theoretical physics, a quantum field theory whose action is given by the Dirichlet energy is known as a sigma model. In such a theory, harmonic maps correspond to instantons.

- One of the original ideas in grid generation methods for computational fluid dynamics and computational physics was to use either conformal or harmonic mapping to generate regular grids.

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